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# On Some Annihilating and Coalescing Systems

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In the present paper we continue the investigation of the so-called coalescing ideal gas in one dimension, initiated by Ermakov. The model consists of point-like particles moving with velocities $\pm 1$ which coalesce and choose a fresh velocity with the same distribution when colliding. In the previous paper various identities in law were derived for the infinitely extended system. In the present paper we consider the scaling limit of the model in its various guises. The main result is the derivation of the scaling limit (invariance principle) for the joint law of an arbitrary finite number of individual particle *trajectories* in this system. We also obtain the scaling limit of the density profile of the system, which strongly resembles earlier results of Belitsky and Ferrari.

**KEY WORDS:** Interacting particle systems; coalescing and annihilating ideal gas; ballistic coalescence and annihilation; random walks; Brownian motion; hydrodynamic limit; invariance principles.

## 1. INTRODUCTION

In the present paper we study the asymptotic behaviour of the time evolution of one-dimensional systems of coalescing/annihilating ballistic particles. The two basic models discussed are the following:

(1) Coalescing Ideal Gas: at $t=0$ at every point of integer coordinate there is a particle. Particles have independent identically distributed *velocities* $v_i$, with distribution $P(v_i = \pm 1) = 1/2$. The time evolution is the following: particles move rectilinearly and uniformly till first collision,

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when two particles collide they coalesce into one single particle, this single particle chooses a new velocity with the same distribution, independently from the full past, and continues flying with this new velocity till the next collision, when the same procedure is repeated. The first author studied this *process* (of infinitely many particles) in ref. 7. The scaling limit of the process (in its various guises) was not treated there.

(2) Annihilating Ideal Gas: the initial conditions are the same, but the time evolution differs. Now, when two particles collide they annihilate. This process was studied by Belitsky and Ferrari in ref. 2, where they prove a scaling limit for the time evolution of the density profile of the system.

The two models can be formulated in a unified way: The particles, besides their i.i.d. $\pm 1$ velocities, also have i.i.d. *masses* $M_i > 0$, $i \in \mathbb{Z}$, which are independent of the velocities. Call $m_i = v_i \cdot M_i$ the *charge* of particle $i$. Now, define the dynamics in the following natural way: when two particles collide they form one new particle with charge equal to the sum of the two incoming charges. I.e.:

$$m^{(out)} = m_1^{(in)} + m_2^{(in)}$$

$$v^{(out)} = \mathrm{sgn}(m^{(out)})$$

$$M^{(out)} = |m^{(out)}| = |M_1^{(in)} - M_2^{(in)}|$$

If ever $m^{(out)} = 0$, then the two particles annihilate. Now, if the initial masses of particles are identically $M_i = 1$ then clearly we get the model of annihilating ideal gas studied by Belitsky and Ferrari. On the other hand, it is easy to see that if initially the masses have i.i.d. exponential distributions of parameter 1 then this model will mimic the coalescing ideal gas. Note, that the dynamics now is strictly deterministic. The randomness of the dynamics formulated in the first paragraph is now encoded in the random initial masses. This construction will be clearly explained in Section 3.

In Section 3 we consider the charge-density profile (or the *profile function*) $\Phi_t$, i.e., the function $x \mapsto$ the total charge between 0 and $x$ at time $t$. It is essentially the same object as the "surface profile" introduced for the annihilating ideal gas by Belitsky and Ferrari.[2] The authors of ref. 2 have found the scaling limit of the surface profile (as $t \to \infty$). We show that the profile function of coalescing ideal gas obeys *the same scaling limit* as that of annihilating ideal gas. This scaling limit can be formulated as an *invariance principle* for the profile function (see Theorem 4 in this paper). This is just a simple remark to the peeper.[2]

We then study in more detail the limiting process of profile functions. We prove, inter alia, that in the scaling limit, the set occupied by particles has Hausdorff dimension 1/2 and the profile function is exactly the distribution function of the "flat" 1/2-Hausdorff measure on this set. These statements are straightforward translations of well-known facts about sample path properties of one-dimensional Brownian motion.

We also study the scaling limit of individual trajectories in the coalescing ideal gas. Note that this question makes sense only in the case of the *coalescing* system: in the annihilating gas individual trajectories die out at the first collision. We prove that the properly rescaled trajectory of a tagged particle in the coalescing ideal gas converges in distribution to the "Brownian flight process" $\eta(\cdot)$ defined as follows:

$$\eta(t) = \int_0^t \operatorname{sgn}(W_s)\, ds$$

where $W$ denotes a one-dimensional Brownian motion started from 0. We also prove *joint* invariance principles for the *trajectories* of any finite number of tagged particles in the system of coalescing ideal gas (note the emphasis on "joint" and "trajectories"). Another result worth mentioning here is the limit law for the rescaled coalescence time of any two particles in the system.

The paper is structured as follows: In Section 2 we reformulate the basic construction of Belitsky and Ferrari. In Section 3 we give precise mathematical meaning to what has been said in this introduction, i.e., we formulate the models of annihilating/coalescing ideal gas, in a joint formalism. In Section 4 we study in detail the limiting object, what we call "Brownian continuous system." In Section 5 we formulate the invariance principle for the rescaled profile functions. In the last three sections we study the particle path properties of coalescing systems. These sections make a *genuinely new* contribution, while Sections 2 to 5 may be considered just as remarks to the paper of Belitsky and Ferrari. In Section 6 we give a general definition of particle paths for a coalescing system and study their properties. In Section 7 we deal with particle paths of the Brownian continuous system, or "coalescing flight processes," a system of uncountably many particles at any time. Finally, in Section 8 we prove the invariance principle for the individual trajectories in coalescing ideal gas.

## Notation

$D(\mathbb{R})$ denotes the set of càdlàg real-valued functions on $\mathbb{R}$. Throughout this paper, we will use only the topology on $D(\mathbb{R})$ induced by uniform convergence on compact intervals. A continuity statement on the space

$D(\mathbb{R})$ without any further detail will mean continuity with respect to this topology.

$C(\mathbb{R})$ denotes the set of continuous real-valued functions on $\mathbb{R}$ (also endowed with the topology induced by uniform convergence on compact intervals).

When $A$ and $B$ are two sets, $T(A, B) := B^A$ will denote the set of mappings of $A$ into $B$. We will for instance use the set $T(\mathbb{R}^+, D(\mathbb{R}))$.

If $\alpha > 0$, then $h_\alpha$ denotes the standard $\alpha$-Hausdorff measure in $\mathbb{R}$ (see e.g., ref. 12 for a precise definition).

## 2. THE DETERMINISTIC SEMI-GROUP

Consider a càdlàg function $\Phi \in D(\mathbb{R})$. For all $t \geqslant 0$, we define the function $S_t(\Phi)$ as follows:

$$S_0(\Phi) = \Phi$$

$$S_t(\Phi)(x) = \inf\{\Phi(x+y) : y \in [-t, t)\} \qquad \text{for all} \quad x \in \mathbb{R}$$

Note that $S_t$ maps $C(\mathbb{R})$ into itself and $D(\mathbb{R})$ into itself.

In the following proposition we list some straightforward properties of $S_t$.

**Proposition 1.** For any $\Phi \in D(\mathbb{R})$:

(i) For all $t \geqslant 0$ and $s \geqslant 0$, $S_t(S_s(\Phi)) = S_{t+s}(\Phi)$. In other words, $(S_t)_{t \geqslant 0}$ is a semi-group of transformations.

(ii) For all $t > 0$, $S_t(\Phi)$ has locally bounded variation. Moreover, there exist two strictly increasing sequences $(x_n)_{n \in \mathbb{Z}}$ and $(y_n)_{n \in \mathbb{Z}}$ such that $\lim_{n \to -\infty} x_n = -\lim_{n \to +\infty} x_n = -\infty$ and for all $n \in \mathbb{Z}$, $x_n < y_n < x_{n+1}$ and $S_t(\Phi)$ is non-decreasing in $[x_n, y_n]$ and non-increasing in $[y_n, x_{n+1}]$.

(iii) For all $a > 0$, $S_t(a\Phi) = aS_t(\Phi)$.

(iv) For all $a > 0$, define $\lambda_a : D(\mathbb{R}) \to D(\mathbb{R})$ by $(\lambda_a \Phi)(x) = \Phi(ax)$. Then

$$S_{at} \circ \lambda_a = \lambda_a \circ S_t$$

We now state some "contractivity" properties of $S.$, which follow immediately from the fact that for $x \in \mathbb{R}$, and for any $\Phi$, $\Psi$ in $D(\mathbb{R})$,

$$|S_t \Phi(x) - S_t \Psi(x)| \leqslant \sup_{y \in [x-t, x+t]} |\Phi(y) - \Psi(y)|$$

**Proposition 2.** (i) Let $T(\mathbb{R}^+, D(\mathbb{R}))$ be the set of trajectories of profile functions endowed with the "uniform uniform" topology induced by uniform convergence on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$.

The mapping $\mathscr{S}: D(\mathbb{R}) \to T(\mathbb{R}^+, D(\mathbb{R}))$ defined by

$$\mathscr{S}(\Phi)(t) = S_t(\Phi)$$

is continuous. In particular, for any fixed $t \geqslant 0$, $S_t: D(\mathbb{R}) \to D(\mathbb{R})$ is continuous.

(ii) For any fixed $\Phi \in C(\mathbb{R})$ the mapping $\mathbb{R}^+ \ni t \mapsto S_t \Phi \in C(\mathbb{R})$ is continuous. (the same is also true for $\Phi \in D(\mathbb{R})$ endowed with Skorokhod topology, which we otherwise do not consider in this paper).

## 3. DISCRETE EXAMPLES

### 3.1. Annihilating Particles

Consider now the following deterministic setting: Define two disjoint locally finite subsets of $\mathbb{R}$: $A_0^+$ and $A_0^-$. Assume that at time 0, at each point of $A_0^+$ (respectively $A_0^-$) a particle starts with unit speed to the right (respectively to the left), and when two particles meet, they annihilate. This type of model has been studied by Fisch,[8] Belitsky-Ferrari[2] and it is closely related to the so-called three-colour cellular automaton. As pointed out in ref. 2 it is very easy to express the positions of living particles at time $t > 0$ using $S_t$.

Define the measure

$$\mu_0 = \sum_{x \in A_0^+} \delta_x - \sum_{x \in A_0^-} \delta_x$$

and the right-continuous function of locally bounded variation $\Phi_0: \mathbb{R} \to \mathbb{R}$, such that $\Phi_0(0) = 0$ and that the derivative $\Phi_0'$ (in the sense of distributions) of $\Phi_0$ is $\mu_0$.

$\Phi_0$ is a step-function with jumps of magnitude $\pm 1$. It is straightforward to check that for any $t > 0$, the function $\Phi_t := S_t(\Phi_0)$ is also a step-function, with jumps of magnitude $\pm 1$. In fact, it is very easy to see that if we define

$$\mu_t = (\Phi_t)'$$

$$A_t = \{x \in \mathbb{R} : \lim_{y \to x-} \Phi_t(y) \neq \lim_{y \to x+} \Phi_t(y)\}$$

$$A_t^- = \{x \in \mathbb{R} : \lim_{y \to x-} \Phi_t(y) = \lim_{y \to x+} \Phi_t(y) + 1\}$$

$$A_t^+ = \{x \in \mathbb{R} : \lim_{y \to x-} \Phi_t(y) = \lim_{y \to x+} \Phi_t(y) - 1\}$$

then $A_t$ (respectively $A_t^+$, $A_t^-$) correspond to the set of particles living at time $t$ (respectively living at time $t$ that move to the right, living at time $t$ that move to the left).

### 3.2. Coalescing Particles

Assume now that we modify the previous model in the following way: Each particle is assigned a (positive or negative) charge and moves to the right or to the left at unit speed according to the sign of their charges: It moves to the right if its charge is positive and it moves to the left if the charge is negative. When two particles of charges $m^+$ and $m^-$ collide, then they stick together and become a single particle of charge $m^+ + m^-$ that moves on with unit speed to the left or to the right depending on the sign of $m^+ + m^-$. Again, once the initial data (the locally finite set of particles $A_0$ with their respective charges $m_x$) is given, this system evolves deterministically. We define this time

$$\mu_0 = \sum_{x \in A_0} m_x \, \delta_x$$

and the function $\Phi_0$ as above. In this case again, the system at time $t$ is described by $\Phi_t := S_t(\Phi_0)$. More precisely, it is easy to check that the set of particles (of non-zero charge) living at time $t$ is the set

$$A_t = \{x \in \mathbb{R} : \lim_{y \to x-} \Phi_t(y) \neq \lim_{y \to x+} \Phi_t(y)\}$$

and that the charge of the particle located at $x \in A_t$ at time $t$ is

$$m_{x,t} = \lim_{\varepsilon \downarrow 0} \{\Phi_t(x+\varepsilon) - \Phi_t(x-\varepsilon)\}$$

An interesting subcase here is the case where the absolute value of the charges of the initial particles are independent identically distributed variables, with an exponential law of parameter 1. Note that when two particles meet, their charges have *different* signs; hence,

- The "outcoming" particle moves to the left (resp. to the right) with probability 1/2.

- The absolute value of the charge of the outcoming particle is again an exponentially distributed random variable of parameter 1, which is independent of the charges and velocities of all other particles living at the same time, and also of its own velocity. Indeed, elementary computations show that if $X$ and $Y$ are two independent random variables with a com-m,on exponential distribution of parameter 1, then $\mathrm{sgn}(X-Y)$ and $|X-Y|$ are independent and have the distributions $P(\mathrm{sgn}(X-Y)=\pm 1)=1/2$, $P(|X-Y|<t)=1-e^{-t}$.

In other words, the law of $(A_t, t \geqslant 0)$ is exactly that corresponding to the positions of coalescing particles moving at unit speed that randomly choose (with probability 1/2-1/2) whether they go to the right or to the left when they collide (and coalesce). This system, with initial state $A_0=\mathbb{Z}$, is also mentioned in Fisch[8] and has been studied in Ermakov,[7] where it was called *coalescing ideal gas*. Let us stress that the system of coalescing particles which randomly choose their direction when they coalesce is not deterministic, but it is equivalent to the deterministic system of particles of i.i.d. exponential randomly signed mass. In the latter deterministic case, all the collision rules are contained in the information provided by the initial data, i.e., the charges of particles living at time 0.

We shall see how this interpretation of the system in terms of $S_t$ provides an economic way of deriving limit results.

## 4. THE BROWNIAN CONTINUOUS SYSTEM

We now briefly study the continuous counterpart of the systems that we just described. This continuous system was introduced in ref. 2. In the next sections, we shall see that it corresponds to the scaling limit of these discrete systems.

Suppose now that $(B_x, x \in \mathbb{R})$ is a two-sided linear Brownian motion with $B_0=0$ (i.e., $(B_x, x \geqslant 0)$ and $(B_{-x}, x \geqslant 0)$ are two independent Brownian motions started from 0). Define then, for all $t \geqslant 0$,

$$\Phi_t = S_t(B)$$

As mentioned in Section 2, $\Phi_t$ is of bounded variation for all $t>0$. In particular, one can define the signed measure $\mu_t=(\Phi_t)'$ (in the sense of distributions) defined on intervals as

$$\mu_t((a, b]) = \Phi_t(b) - \Phi_t(a)$$

This measure $\mu_t$ can be loosely speaking interpreted as a regularisation of the white noise.

Let us stress again that the only random part comes from the initial data $\Phi_0=B$, and that the evolution of $\Phi_t$ given $\Phi_0$ is then deterministic.

We now state some results that give some insight into the process $(\Phi_t, t \geqslant 0)$. As $\mu_t$ is a signed measure (when $t>0$), it is the difference of two non-negative measures $\mu_t^+$ and $\mu_t^-$ so that $\mu_t = \mu_t^+ - \mu_t^-$.

We now briefly recall the definition of the 1/2-Hausdorff measure (see e.g., ref. 12, Section 10 for details). Let $A \subset \mathbb{R}$ denote a bounded set, and for any $\varepsilon > 0$, and any covering of $A$ by intervals $(A_n)$ of respective length $a_n$ such that $a_n < \varepsilon$, consider the sum $\sum_n \sqrt{a_n}$. The infimum (when $\varepsilon > 0$ is fixed) of these sums (the infimum is taken over all such coverings) is denoted by $h_{1/2}^{\varepsilon}(A)$. When $\varepsilon \to 0$, $h_{1/2}^{\varepsilon}(A)$ increases to some (possibly infinite) value $h_{1/2}(A)$ called the 1/2-Hausdorff measure of $A$. In the special case where $h_{1/2}(A) \in (0, +\infty)$, then the Hausdorff dimension of $A$ is 1/2. If $A \subset \mathbb{R}$ is unbounded and if for all large enough $x$, the Hausdorff dimension of $A \cap [-x, x]$ is 1/2, then one says that the Hausdorff dimension of $A$ is also 1/2.

**Theorem 3.** (i) For all $t > 0$, the supports $\Lambda_t$, $\Lambda_t^+$ and $\Lambda_t^-$ of $\mu_t$, $\mu_t^-$ and $\mu_t^+$ are sets of Hausdorff dimension 1/2, and of locally finite 1/2-Hausdorff measure.

(ii) The measure $\mu_t^+$ (resp. $\mu_t^-$) is exactly the 1/2-Hausdorff measure supported by the set $\Lambda_t^+$ (resp. $\Lambda_t^-$). In other words, if the interval $I = (a, b)$ does not intersect $\Lambda_t^-$ (i.e., that $\Phi_t$ is non-decreasing on $I$), then $\Phi_t(b) - \Phi_t(a)$ is precisely the 1/2-Hausdorff measure of $\Lambda_t^+ \cap I$.

The sets $\Lambda_t^+$ and $\Lambda_t^-$ should be interpreted as the sets of particles moving to the right and to the left at time $t$. As opposed to the previous discrete cases, these sets are uncountable, and their "mass" is measured by the 1/2-Hausdorff measure.

*Proof of Theorem 3.* Let us first recall some well-known facts concerning the level-sets of Brownian motion. Define the one-dimensional Brownian motion $(W_x, x \geqslant 0)$ started from 0 and denote its local time at level 0 and time $x$ by $\ell_x$. Define also for all $x \geqslant 0$,

$$W_x^* = \sup_{y \in [0, x]} W_y$$

The law of $((W_x^* - W_x, W_x^*), x \geqslant 0)$ is identical to that of $((|W_x|, \ell_x), x \geqslant 0)$ (see e.g., ref. 14, VI. (2.3)). On the other hand, the local time at 0 of Brownian motion can be exactly defined as the 1/2-Hausdorff measure of the set of zeros of this Brownian motion (see ref. 11, Section 2.5); more precisely, for all $x \geqslant 0$,

$$\ell_x = h_{1/2}(\{ y \in [0, x] : W_y = 0 \})$$

Combining these two facts shows immediately that

$$(h_{1/2}(\{y \in [0, x]: W_y = W_y^*\}), x \geq 0) \stackrel{d}{=} (W_x^*, x \geq 0) \quad (1)$$

Let us now come back to the actual proof of Theorem 3: We now say that $x$ is a point of right-increase (resp. left-increase) of $\Phi_t$ if and only if there exists $\varepsilon > 0$ such that such that for all $y \in (x, x+\varepsilon)$ (resp. $y \in (x-\varepsilon, x)$) $\Phi_t(y) > \Phi_t(x)$ (resp. $\Phi_t(y) < \Phi_t(x)$). Note that points of right-increase and left-increase play here a different role due to the non-symmetry of the definition of $S_t$ (we used inf and not sup).

Clearly, the definition of $\Phi_t$ implies that $x \in \mathbb{R}$ is a point of right-increase of $\Phi_t$ if and only if, for all $y \in (x-t, x+t]$, $B_y > B_{x-t}$.

Let us now define the set

$$H_t = \{x \in \mathbb{R} : x \text{ is a point of right-increase of } \Phi_t\}$$

We are now going to show that $H_t$ and $\Lambda_t^+$ differ by at most countably many points. Clearly, $H_t \subset \Lambda_t^+$. Suppose for a moment that $x \in \Lambda_t^+ \backslash H_t$. As $x \notin H_t$, there exists $y \in (x-t, x+t]$ such that $B_y \leq B_{x-t}$. As $x \in \Lambda_t^+ \backslash H_t$, it is a point of left-increase of $\Phi_t$, and this implies readily that $B_{x-t} = \Phi_t(x)$. Hence one of the following two events is necessarily true:

- $B_{x-t} = B_{x+t}$ (in other words $y = x+t$).
- $y$ is a local minimum of $B$ (this happens if $y \neq x+t$).

Note now that for any rational number $q \in \mathbb{Q}$, there can exist only one $x \in (q-t, q+t)$ such that $B_{x-t} = B_{x+t}$ and for all $z \in (x-t, x+t)$, $B_z \geq B_{x-t}$. Hence, almost surely, for all $t > 0$,

$$\{x \in H_t \backslash \Lambda_t^+ : B_{x-t} = B_{x+t}\} \text{ is at most countable} \quad (2)$$

Let us now consider the case where $y$ is a local minimum of $B$. Note that any two local minima of $B$ do occur at different heights (as $B$ only countably many local minima). Hence,

$$x - t = \sup\{z < y : B_z = B_y\}$$

In other words, there exists a surjection of the set of local minima of $B$ onto the set

$$\{x \in H_t \backslash \Lambda_t^+ : \exists y \in (x-t, x+t), B_{x-t} = B_y = \Phi_t(x)\}$$

which is therefore also at most countable.

Finally, putting the pieces together (using (2)), we get that almost surely, for all $t > 0$,

$$H_t \backslash \Lambda_t^+ \text{ is at most countable} \tag{3}$$

Suppose now for a moment that $x \in H_t$. In particular, this implies (by continuity of $B$ and because for all $y \in (x-t, x+t]$, $B_y > B_{x-t}$) that there exists a rational $q > x+t$, such that

$$x - t = \sup\{ y \leqslant q : B_y \leqslant B_{x-t} \}$$

Similarly, for all $z \in (x-t, x)$ that is a point of right-increase of $\Phi_t$, the previous observation yields readily that

$$z - t = \sup\{ y \leqslant x - t : B_y = B_{z-t} \}$$

But as for all $y \in (x-t, q]$, $B_y > B_{x-t} > B_{z-t}$, we get

$$z - t = \sup\{ y \leqslant q : B_y = B_{z-t} \}$$

Hence, $H_t \cap (x-t, x)$ corresponds exactly to hitting times of its maximum of the reversed process started at $q$. More precisely,

$$H_t \cap (x-t, x) = \{ y + t \in (x-t, x) : W^q_{q-y} = \sup_{[0, q-y]} W^q(\cdot) \}$$

where $W^q(\cdot) = B(q) - B(q - \cdot)$. Hence, combining this with (1) implies that for all $a \in (x-t, x)$,

$$h_{1/2}(H_t \cap (a, x)) = B_{x-t} - \inf_{y \in [a-t, x-t]} B_y = \Phi_t(x) - \min_{[a, x]} \Phi_t(\cdot)$$

This implies (using Proposition 1(ii) and (3)) all results dealing with $\Lambda_t^+$ stated in the Theorem. Those concerning $\Lambda_t^-$ are derived via a symmetry argument. ∎

## 5. INVARIANCE PRINCIPLE FOR THE PROFILE FUNCTION

Given a discrete (annihilating or coalescing) particle system started from the integer lattice points, as described in Section 3, denote by $\hat{\Phi}_0(x)$ its profile function, i.e., the total charge in the interval between the origin and the point of coordinate $x$, at time 0:

$$\hat{\Phi}_0(x) := \begin{cases} \sum_{i \in \mathbb{Z} \cap (0, x]} m_i & \text{if} \quad x > 0 \\ -\sum_{i \in \mathbb{Z} \cap (x, 0]} m_i & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \end{cases} \tag{4}$$

where the $m_i$ are i.i.d. charges with $E(m_i) = 0$ and $E(m_i^2) = 1$. We now introduce the time-evolved profile

$$\hat{\Phi}_t = S_t(\hat{\Phi}_0)$$

that is the profile function at time $t$.

The formalism introduced in the previous sections helps us to formulate and immediately prove functional limit theorems (invariance principles) for the resealed profile function $\hat{\Phi}^{(N)}$ defined as follows:

$$\hat{\Phi}_t^{(N)}(x) := N^{-1/2} \hat{\Phi}_{Nt}(Nx)$$

Using Proposition 1, (iii) and (iv), it is straightforward to check that

$$\hat{\Phi}_t^{(N)} = S_t \hat{\Phi}_0^{(N)}$$

Hence we directly conclude the following functional limit theorem for the profile function evolving in time:

**Theorem 4.** The sequence $(\hat{\Phi}_\cdot^{(N)}(\cdot)) = \mathscr{S}(\hat{\Phi}^{(N)})$ converges weakly in $T(\mathbb{R}^+, D(\mathbb{R}))$ (endowed with the "uniform uniform topology") to $\mathscr{S}(B)$ when $N \to \infty$, where $B$ is a two-sided Brownian motion with $B_0 = 0$.

Indeed: This theorem follows directly from the weak convergence $\hat{\Phi}_0^{(N)} \Rightarrow B$ in $D(\mathbb{R})$ and the continuity of $\mathscr{S}$: $D(\mathbb{R}) \to T(\mathbb{R}^+, D(\mathbb{R}))$.

## 6. BROWNIAN FLIGHT PROCESS. PARTICLE PATHS IN COALESCING SYSTEMS

From now on we shall concentrate on the systems of coalescing particles. More precisely, we are going to study the scaling limit of trajectories of *individual* particles in the coalescing ideal gas. For this purpose we introduce

**Definition 5.** The (Brownian) flight process is

$$\eta(t) = \int_0^t \operatorname{sgn}(W_s)\, ds, \qquad t \in \mathbb{R}^+ \tag{5}$$

Here, as earlier, $W.$ is a one-dimensional Brownian motion started from 0.

This process consists of countably many linear segments with slopes $\pm 1$ ("flights"). The length of each segment is distributed as the length of an excursion of Brownian motion.

**Proposition 6.** (i) The flight process is self-similar in distribution

$$\frac{\eta(\alpha\cdot)}{\alpha} \overset{d}{=} \eta(\cdot), \qquad \forall \alpha \in \mathbb{R}^+ \setminus \{0\}$$

(ii) The density of $\eta(t)$ on $(-t, t)$ is

$$\frac{dy}{\pi \sqrt{t^2 - y^2}}$$

*Proof.* (i) follows from the self-similarity of Brownian motion, and (ii) is a direct consequence of the Arcsine law of $\int_0^1 1_{\{W_s > 0\}}\, ds$ (see e.g., ref. 14, p. 255). ∎

The Brownian flight process is important for us, because, as we shall see below in (7), it is the scaling limit of a trajectory of a coalescing ideal gas particle.

Consider the coalescing ideal gas system (CIG), as described in Section 3.2, with the initial particle set $A_0 = \mathbb{Z}$, their masses $M_x$ distributed exponentially with parameter 1, and velocities $v_x$ equal to $+1$ or $-1$ with probability 1/2. Let $\hat{\Phi}_0(x)$ be the corresponding profile function, as defined in (4). Let us denote by $\{\hat{\eta}_x(t)\}_{t \in \mathbb{R}^+}$ the path of the particle which starts at $x \in \mathbb{Z}$. The motion of the particle follows the motion of the corresponding discontinuity of $\hat{\Phi}_t = S_t \hat{\Phi}_0$.

From Theorem 1(i) and Lemma 3(ii) in ref. 7 it follows immediately that, for the CIG with initial particle set $\mathbb{Z}$, a particle trajectory starting from $x \in \mathbb{Z}$ can be expressed by

$$\{\hat{\eta}_x(n)\}_{n \in \mathbb{Z}^+} \overset{d}{=} \left\{ x + \frac{1}{4} \sum_{k=0}^{4n-1} \operatorname{sgn}(s_k + s_{k+1}) \right\}_{n \in \mathbb{Z}^+} \tag{6}$$

where $(s_n, n \geqslant 0)$ is a simple symmetric random walk started from $s_0 = 0$. Note that in the setup of ref. 7 the space and time coordinates are twice as large as here. The factor $\frac{1}{4}$ in (6) can intuitively be justified by the observation that a CIG particle trajectory can change its direction at any half-integer time, while $\operatorname{sgn}(s_k + s_{k+1})$ can change only at an even $k$.

By Donsker's theorem, (6) implies the scaling limit result

$$\frac{\hat{\eta}_{nx}(n\cdot)}{n} \underset{n \to \infty}{\Rightarrow} x + \eta(\cdot) \qquad \text{in} \quad C(\mathbb{R}^+), \qquad \text{for any} \quad x \in \mathbb{Z} \tag{7}$$

Now, we would like to generalise (7) to deal with *joint distributions of finitely many trajectories*, and in particular we want to compute the scaling limit of the collision time of two particles. But the simple representation (6) works only for one particle path. So we have to *construct multiple particle paths on the same probability space*. This can be done, for both "discrete" and "continuous" coalescing systems, by using the profile function machinery of Section 3.2 as follows.

Let $\Phi_0(\cdot) \in D(\mathbb{R})$ be an arbitrary initial underlying profile function. Further on we shall leave out the index 0 and write it simply as $\Phi(\cdot)$. We assume the following natural condition:

$$\liminf_{x \to +\infty} \Phi(x) = \liminf_{x \to -\infty} \Phi(x) = -\infty$$

As we shall see in Proposition 7 (see also Fig. 2), this assumption in fact implies that any two particles will eventually collide. One could (with very slight modifications) adapt the following construction to a general $\Phi$.

We shall use the notation

$$f(x \pm) = \lim_{\varepsilon \downarrow 0} f(x \pm \varepsilon)$$

for any càdlàg function $f$.

We say that a particle starts from $x \in \mathbb{R}$ if $\Phi(\cdot)$ is not constant in the neighbourhood of $x$. Let us denote by $\{\xi(\Phi, x, t)\}_{t \in \mathbb{R}^+}$ the trajectory of the *tagged particle* which starts at such a point $x$. It is defined as follows. Let $h \in \mathbb{R}$ be an auxiliary variable.

$$l_x(h) = \sup\{y < x : \Phi(y) < h\}$$

$$r_x(h) = \inf\{y > x : \Phi(y) < h\}$$

$$\theta_x(h) = \frac{1}{2}(r_x(h) - l_x(h))$$

$$\chi_x(h) = \frac{1}{2}(r_x(h) + l_x(h))$$

$$h^* = h^*(\Phi, x, t) = \sup\{h \in \mathbb{R} : \theta_x(h) > t\}$$

$$\xi_x(t) = \begin{cases} \chi_x(h^*) & \text{if} \quad \theta_x(h^*) = \theta_x(h^*+) \\ \chi_x(h^*+) + \dfrac{\chi_x(h^*) - \chi_x(h^*+)}{\theta_x(h^*) - \theta_x(h^*+)}(t - \theta_x(h^*+)) & \\ \qquad \text{if} \quad \theta_x(h^*) \neq \theta_x(h^*+). \end{cases} \tag{8}$$

This construction is illustrated by Figs. 1 and 2.

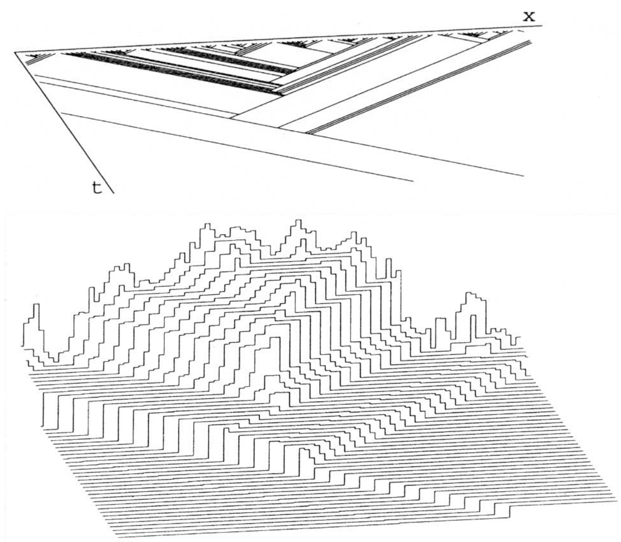


Fig. 1. Coalescing ideal gas trajectories $\hat{\eta}_x(t) = \xi(\hat{\Phi}_0, x, t)$ and the underlying profile functions $\hat{\Phi}_t(x) = (S_t \hat{\Phi}_0)(x)$.

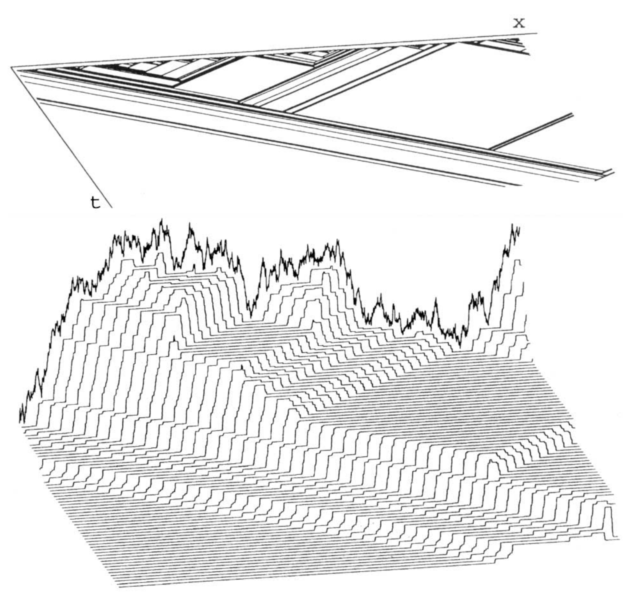


Fig. 2. Coalescing flight processes $\eta_x(t) = \xi(B, x, t)$ and the underlying profile functions $\Phi_t(x) = (S_t B)(x)$.

Note that $\xi$ is invariant under scaling of $\Phi$: For all $\lambda > 0$, $x \in \mathbb{R}$ and $t \geq 0$,

$$\xi(\lambda\Phi, x, t) = \xi(\Phi, x, t) \tag{9}$$

It is straightforward to see that $h^*$ is the minimum of the underlying profile function $\Phi_t$ in the vicinity of the tagged particle position at time $t$: $h^* = \min\{\Phi_t(\xi(\Phi, x, t)), \Phi_t(\xi(\Phi, x, t)-)\}$. $\theta_x(h^*)$ and $\chi_x(h^*)$ are the time and the spatial location of the "first" collision of the tagged particle at or after time $t$. (For Brownian continuous system considered in the next section, "first" means the time-infimum). $l_x(h^*)$ and $r_x(h^*)$ are the starting points of the two particles which take part in this collision and which did not change their direction of movement before it.

Note that $h \mapsto l_x(h)$ is non-decreasing and that $h \mapsto r_x(h)$ is non-increasing. Therefore $\theta_x(\cdot)$ is non-increasing, and for any $\Phi \in D(\mathbb{R})$ and $x \in \mathbb{R}$

$$|\theta_x(h) - \theta_x(h')| \geq |\chi_x(h) - \chi_x(h')| \qquad \forall h, h' \in \mathbb{R} \tag{10}$$

From this it is clear that all trajectories $\xi(\Phi, x, \cdot)$ are Lipschitz-continuous of order 1:

$$|\xi_x(t) - \xi_x(t')| \leq |t - t'| \qquad \forall \Phi \in D(\mathbb{R}), \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+ \tag{11}$$

Note however that some particles can move with speed slower than 1: For instance if $x$ is a local minimum of $\Phi$, then $|\xi'_x(0)| < 1$.

Suppose that $\Phi(\cdot)$ is not constant in the neighbourhoods of $x$ and $y \in \mathbb{R}$, $x < y$. Let us define the *coalescence time* $T_{x,y}(\Phi)$ by the formulae:

$$\begin{aligned} g_{x,y} &= \inf\{\Phi(z), z \in (x, y)\} \\ T_{x,y} &= T_{x,y}(\Phi) = \theta_x(g_{x,y}) \end{aligned} \tag{12}$$

The term "coalescence time" is explained by

**Proposition 7.** Let $x$ and $y$ be as in the above definition. Then the particles which start from $x$ and $y$ coalesce at time $T_{x,y}(\Phi)$, i.e.,

$$\begin{aligned} \xi(\Phi, x, t) &\neq \xi(\Phi, y, t) \qquad \text{if} \quad t < T_{x,y}(\Phi) \\ \xi(\Phi, x, t) &= \xi(\Phi, y, t) \qquad \text{if} \quad t \geq T_{x,y}(\Phi) \end{aligned}$$

First we shall prove a technical lemma.

**Lemma 8.** For all $h \in \mathbb{R}$, for all $x \in \mathbb{R}$,

$$\xi_z(\theta_x(h)) = \chi_x(h) \qquad \forall z \in [l_x(h), r_x(h)] \tag{13}$$

Moreover, if $l_x(h+) < r_x(h+)$ then

$$\xi_z(\theta_x(h+)) = \chi_x(h+) \qquad \forall z \in (l_x(h+), r_x(h+)) \tag{14}$$

***Proof of Lemma 8.*** By the definition, $l.(h)$, $r.(h)$, $\theta.(h)$ and $\chi.(h)$ are constant in the interval $[l_x(h), r_x(h)]$. (13) follows directly. (14) follows from (13) and the left-continuity of $l_x(\cdot)$, $r_x(\cdot)$, $\theta_x(\cdot)$ and $\chi_x(\cdot)$. ∎

***Proof of Proposition 7.*** (i) First we consider the case $t = T_{x,y}$. Since $x, y \in [l_x(g_{x,y}), r_x(g_{x,y})]$, by (13) we have $\xi_x(T_{x,y}) = \xi_y(T_{x,y})$.

(ii) Now we consider the case $t > T_{x,y}$. It follows directly from the definitions, that

$$h^*(\Phi, x, t) < h^*(\Phi, x, T_{x,y}) \leqslant g_{x,y}$$
$$\theta_x(h) = \theta_y(h) \qquad \forall h \leqslant g_{x,y}$$
$$\chi_x(h) = \chi_y(h) \qquad \forall h \leqslant g_{x,y}$$

By (8) it follows that $\xi_x(t) = \xi_y(t)$.

(iii) And finally we consider the case $t < T_{x,y}$. Let us assume first that $h^*(\Phi, x, 0) > g_{x,y}$ and $h^*(\Phi, y, 0) > g_{x,y}$; (Recall that $h^*(\Phi, z, 0) = \min\{\Phi(z), \Phi(z-)\}$). By (14) we have $\xi_x(\theta_x(g_{x,y}+)) = \chi_x(g_{x,y}+)$. By the Lipschitz-continuity (11), this means that

$$\xi_x(t) \leqslant \chi_x(g_{x,y}+) + (\theta_x(g_{x,y}+) - t)$$
$$= r_x(g_{x,y}+) - t, \qquad \forall t \in [0, \theta_x(g_{x,y}+)]$$

On the other hand, the assumption $h^*(\Phi, y, 0) > g_{x,y}$ implies that $y > r_x(g_{x,y}+)$, and hence

$$\xi_y(t) > r_x(g_{x,y}+) - t, \qquad \forall t \in \mathbb{R}^+$$

This proves that $\xi_x(t) < \xi_y(t)$, $t \in [0, \theta_x(g_{x,y}+)]$. An analogous argument shows that the same inequality holds for $t \in [0, \theta_y(g_{x,y}+)]$. In the remaining interval $[\max\{\theta_x(g_{x,y}+), \theta_y(g_{x,y}+)\}, T_{x,y}]$ both particles move along non-parallel rectilinear paths, which start at two different points and collide at time $T_{x,y}$, according to (i). Therefore, the two paths do not intersect before $T_{x,y}$.

In the case $h^*(\Phi, x, 0) = g_{x,y} : \theta_x(g_{x,y}+) = 0$, $\chi_x(g_{x,y}+) = x$, and the particle moves rectilinearly from time 0 to $T_{x,y}$, which makes the proof just simpler. The same holds if $h^*(\Phi, y, 0) = g_{x,y}$. ∎

The definition (8) works correctly for the discrete CIG (see Fig. 1 in the end of the article):

**Proposition 9.** The CIG particle trajectories $\hat{\eta}_x(t)$ defined in the beginning of this section can be expressed in terms of the underlying profile function $\hat{\Phi} = \hat{\Phi}_0$ in the following way:

$$\hat{\eta}_x(t) = \xi(\hat{\Phi}, x, t), \; x \in \mathbb{Z}, \; t \in \mathbb{R}^+ \tag{15}$$

The proof is straightforward, since in Section 3.2 it is shown that the profile function $\hat{\Phi}$ indeed corresponds to the CIG.

## 7. COALESCING FLIGHT PROCESSES

The particle path construction of the previous section (8) can also be applied to the Brownian continuous system: In this case we define the particle trajectories by

$$\eta_x(t) = \xi(B, x, t), \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+ \tag{16}$$

where $\mathbb{R} \ni x \mapsto B_x$ is the Brownian profile function, i.e., a two-sided Brownian motion, just as in Section 4. We call the family $\{\eta_x(\cdot)\}_{x \in \mathbb{R}}$ a *system of coalescing flight processes* (in short: CFP). This name is justified by Propositions 7 and 10.

**Proposition 10.** (i) The CFP particle trajectories are equidistributed with the Flight Process: For all fixed $x \in \mathbb{R}$,

$$\eta_x(\cdot) \stackrel{d}{=} x + \eta(\cdot) \tag{17}$$

(ii) CFP is self-similar in distribution, as space and time scale by the same factor: For all $\alpha > 0$,

$$\left\{\frac{\eta_{\alpha x}(\alpha t)}{\alpha}\right\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} \stackrel{d}{=} \{\eta_x(t)\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} \tag{18}$$

*Proof.* (i) Without loss of generality we assume $x=0$. Define the sets:

$$M_R = \{\alpha \geqslant 0 : (\exists \varepsilon > 0) \text{ such that } B_\alpha = \inf\{B_z : z \in [0, \alpha+\varepsilon]\}\}$$

$$M_L = \{\alpha \leqslant 0 : (\exists \varepsilon > 0) \text{ such that } B_\alpha = \inf\{B_z : z \in [\alpha-\varepsilon, 0]\}\}$$

The sets $M_R$ and $M_L$ are countable, so we can list their elements as

$$M_R = \{\alpha_i : i = 1, 2, 3,...\} \qquad M_L = \{\alpha_i : i = -1, -2, -3,...\}$$

For convenience we shall denote $\alpha_0 = 0$. Denote also:

$$h_i = B_{\alpha_i}, \qquad i \in \mathbb{Z}$$

$$\beta_i = \begin{cases} \inf\{z > \alpha_i : B_z = h_i\}, & \text{if } i \geqslant 0 \\ \sup\{z < \alpha_i : B_z = h_i\}, & \text{if } i < 0 \end{cases}$$

$$\gamma_i = \begin{cases} \sup\{z \leqslant 0 : B_z = h_i\}, & \text{if } i \geqslant 0 \\ \inf\{z \geqslant 0 : B_z = h_i\}, & \text{if } i < 0 \end{cases}$$

$$\rho_i = |\alpha_i - \gamma_i|, \qquad i \in \mathbb{Z}$$

$$\sigma_i = |\beta_i - \gamma_i|, \qquad i \in \mathbb{Z}$$

Note that by the above definitions we have

$$\gamma_i < 0 < \alpha_i < \beta_i \qquad \text{for} \quad i > 0$$

$$\alpha_0 = \beta_0 = \gamma_0 = 0$$

$$\beta_i < \alpha_i < 0 < \gamma_i \qquad \text{for} \quad i < 0$$

The following lemma states well known pathwise properties of Brownian motion:

**Lemma 11.** For almost all Brownian profile functions $B.$,

(i) $\{h_i : i \in \mathbb{Z}\}$ is dense in $(-\infty, 0]$,

(ii) for any $i, j \in \mathbb{Z}$ with $i \neq j$ one of the following two alternatives holds:

- either $h_i < h_j$ and in this case $\sigma_j < \rho_i$,
- or $h_j < h_i$ and in this case $\sigma_i < \sigma_j$.

Indeed, (i) follows from the fact that a.s. the heights of the local minima of Brownian motion form a dense set in $\mathbb{R}$; (ii) is a consequence of the fact that a.s. there are no two local extrema of the same hight.

From this simple lemma it immediately follows that the closed intervals $[\rho_i, \sigma_i]$, $i \in \mathbb{Z}$ are pairwise disjoint and their union is dense in $\mathbb{R}^+$. Given this fact we can define the function $\mathbb{R}^+ \ni s \mapsto X_s \in \mathbb{R}$ as follows:

$$X_s = \begin{cases} B_{s+\gamma_i} - h_i & \text{if } s \in [\rho_i, \sigma_i] \quad \text{for some } i \geqslant 0 \\ -B_{-s+\gamma_i} + h_i & \text{if } s \in [\rho_i, \sigma_i] \quad \text{for some } i < 0 \\ 0 & \text{if } s \notin \bigcup_{i \in \mathbf{Z}} [\rho_i, \sigma_i] \end{cases}$$

In plain words this definition means the following: we take the two independent (one-sided) Brownian paths $B_s$, $s \geqslant 0$ and $B'_s = B_{-s}$, $s \geqslant 0$ and we define the processes

$$R_t = B_t - \min_{0 \leqslant s \leqslant t} B_s, \qquad R'_t = B'_t - \min_{0 \leqslant s \leqslant t} B'_s$$

It is well known that $R_.$ and $R'_.$ defined this way will be two independent Brownian motions reflecting From $0.X_.$ is obtained by "gluing together $R_.$ and $-R'_.$ according to their local time at 0," i.e., the excursions of $R_.$ and $-R'_.$ away From 0 are glued together according to the a.s. well defined order of their occurrence. $X_.$ obtained this way is an other Brownian motion. Finally, it is straightforward to see that in case of a Brownian profile function (which is a.s. continuous, has no points of increase or decrease and has no two local extrema of the same height) the definition (15) of $\eta_0(t)$ is equivalent to

$$\eta_0(t) = \int_0^t \operatorname{sgn}(X_s)\, ds$$

(ii) Self-similarity follows From the self-similarity of Brownian motion and of $\xi$ (9), and From (16):

$$\begin{aligned} \left\{\frac{\eta_{\alpha x}(\alpha t)}{\alpha}\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} &= \left\{\frac{\xi(B, \alpha x, \alpha t)}{\alpha}\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} \\ &\overset{d}{=} \left\{\frac{\xi(x \mapsto \alpha^{-1/2} B_{\alpha x}, \alpha x, \alpha t)}{\alpha}\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} \\ &= \left\{\xi(\alpha^{-1/2} B, x, t)\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} \\ &= \left\{\xi(B, x, t)\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} = \left\{\eta_x(t)\right\}_{x \in \mathbf{R},\, t \in \mathbf{R}^+} \qquad \blacksquare \end{aligned}$$

Now we can compute the law of the coalescence time $T_{x,y} = T_{x,y}(B)$ of two tagged CFP particles (12): Let $Z$ denote a standard stable random variable of index 1/2 (i.e., that has the same law as the hitting time of 1 by a standard Brownian motion started from 0). Let $W$ denote a Brownian motion with $W_0 = 0$, that is independent of $Z$, and define $I_1 = \inf_{s \in [0,1]} W_s$.

**Proposition 12.** For the CFP system, for any $x, y \in \mathbb{R} : x < y$,

$$\frac{T_{x,y}}{y-x} \stackrel{d}{=} T_{0,1} \stackrel{d}{=} \frac{1}{2}(1 + (W_1 - 2I_1)^2 Z) \tag{19}$$

In other words, $2T_{0,1} - 1$ is distributed with the density

$$\frac{2t^{1/2}}{\pi(t+1)^2} \qquad \text{on} \qquad \mathbb{R}^+ \tag{20}$$

*Proof.* The first relation in (19) is clear From the self-similarity of CFP (18). From the definition of $T_{x,y}$ (12) it is clear that

$$T_{0,1} = \tfrac{1}{2}(\sup\{s \leqslant 0 : B_s \leqslant I_1\} + \inf\{s \geqslant 1 : B_s \leqslant I_1\})$$

Conditionally on $\{B_s, s \in [0,1]\}$, $\sup\{s \leqslant 0 : B_s \leqslant I_1\}$ and $\inf\{s \geqslant 1 : B_s \leqslant I_1\} - 1$ are independent and their laws are respectively identical to that of $I_1^2 Z$ (i.e., the hitting time of $I_1$ by $x \mapsto B_{-x}$) and $(B_1 - I_1)^2 Z$ (i.e., the hitting time of $I_1 - B_1$ by $x \mapsto B_{1+x} - B_1$). Hence, as $I_1 \leqslant 0$ and $I_1 \leqslant B_1$, we get (using the fact that $Z$ is stable of order 1/2) that

$$T_{0,1} \stackrel{d}{=} \tfrac{1}{2}(1 + (B_1 - 2I_1)^2 Z)$$

Let us compute the density of $(B_1 - 2I_1)^2 Z$. By integrating the joint density of $(B_1, I_1)$, $\sqrt{2/\pi}(a-2b)\, e^{-(a-2b)^2/2}$ on $\{(a,b): a \geqslant b, b \leqslant 0\}$ (see ref. 14, p. 105) we obtain

$$\begin{aligned}
&\frac{P(B_1 - 2I_1 \in [\alpha, \alpha + d\alpha])}{d\alpha} \\
&\quad = \frac{1}{d\alpha} \int_{-\alpha}^{0} P(I_1 \in [b, b+db], B_1 \in [2I_1 + \alpha, 2I_1 + \alpha + d\alpha])\, db \\
&\quad = \int_{-\alpha}^{0} \sqrt{\frac{2}{\pi}} (2b + \alpha - 2b)\, e^{-(2b+\alpha-2b)^2/2}\, db \\
&\quad = \sqrt{\frac{2}{\pi}} \alpha e^{-\alpha^2/2} \int_{-\alpha}^{0} db = \sqrt{\frac{2}{\pi}} \alpha^2 e^{-\alpha^2/2}
\end{aligned}$$

It is well-known that

$$\frac{P(Z \in [t, t+dt])}{dt} = \frac{1}{\sqrt{2\pi t^3}} e^{-1/2t}$$

(see ref. 6, p. 353). Therefore, one easily gets (20). ∎

## 8. INVARIANCE PRINCIPLE FOR PARTICLE PATHS

The following theorem shows that, under the natural scaling (18), any finite set of CIG trajectories converges weakly to the corresponding CFP trajectories:

**Theorem 13.** (Invariance principle for particle paths). For any finite set $X = \{x_1,..., x_k\} \subset \mathbb{R}$,

$$\left\{\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n}\right\}_{x \in X,\, t \in \mathbb{R}^+} \underset{n \to \infty}{\Rightarrow} \{\eta_x(t)\}_{x \in X,\, t \in \mathbb{R}^+}$$

in the topology of uniform convergence on compacts on $(C(\mathbb{R}^+))^k$.

First we state the $B$-a.s. continuity of a CFP particle path $D(\mathbb{R}) \ni B \mapsto \xi(B, x, \cdot)$ in $T(\mathbb{R}, C(\mathbb{R}^+))$:

**Lemma 14.** For almost all Brownian profiles $B$., for all $T > 0$, for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exist $\delta = \delta(B, x, \varepsilon, T) > 0$, $L = L(B, x, \varepsilon, T) \in (-\infty, x)$ and $R = R(B, x, \varepsilon, T) \in (x, \infty)$ such that for any profile function $\tilde{B}. \in D(\mathbb{R})$ if $\sup\{|\tilde{B}_z - B_z| : L \leqslant z \leqslant R\} < \delta$ then $\sup\{|\xi(\tilde{B}, x, t) - \xi(B, x, t)| : 0 \leqslant t \leqslant T\} < \varepsilon$.

*Proof of Lemma 14.* In order to simplify notation we shall denote $\xi_x(t) = \xi(B, x, t)$ and $\tilde{\xi}_x(t) = \xi(\tilde{B}, x, t)$. Note first that due to Lipschitz continuity of $t \mapsto \xi(\tilde{B}, x, t)$ (11) for any initial position $x$ and profile function $\tilde{B}$, it suffices to prove $|\xi(\tilde{B}, x, t) - \xi(B, x, t)| < \varepsilon$ for any $t \in \mathbb{R}^+$ fixed. We shall prove it for $t = 1$. Let the Brownian profile $B$. and the initial position $x \in \mathbb{R}$ be fixed. We shall exploit the following almost sure properties of the Brownian profile function: (1) $B$ is a.s. continuous; (2) $B$ a.s. does not have two or more local extrema at the same hight; (3) almost surely $t = 1$ is not a collision time of the trajectory $\xi_x(t)$, i.e., $\theta^- < 1 < \theta^+$, where $\theta^{\pm}$ are defined below.

Denote

$$h^* = h^*(B, x, 1) = \inf\{h\colon \theta_x(h) < 1\} = \sup\{h\colon \theta_x(h) > 1\}$$

$$l^\pm = l_x(h^* \mp) = \lim_{\varepsilon \downarrow 0} l_x(h^* \mp \varepsilon),$$

$$r^\pm = r_x(h^* \mp) = \lim_{\varepsilon \downarrow 0} r_x(h^* \mp \varepsilon),$$

$$\theta^\pm = \tfrac{1}{2}(r^\pm - l^\pm)$$

$$\chi^\pm = \tfrac{1}{2}(r^\pm + l^\pm)$$

Note that due to the continuity of $B_\cdot$, we have

$$h^* = B_{l^\pm} = B_{r^\pm}$$

From the definition of the particle trajectories (8) it follows that

$$\xi_x(\theta^\pm) = \chi^\pm, \qquad \xi_x(1) = \frac{\theta^+ - 1}{\theta^+ - \theta^-}\chi^- + \frac{1 - \theta^-}{\theta^+ - \theta^-}\chi^+$$

From the fact that a.s. there are no two local minima of the same hight, it follows that almost surely one of the following two alternatives holds: *either* $l^+ = l^- < x < r^- < r^+$, in which case $l^\pm$ and $r^+$ are not local extrema and $r^-$ is a local minimum of $B$; *or* $l^+ < l^- < x < r^- = r^+$, in which case $r^\pm$ and $l^+$ are not local extrema and $l^-$ is a local minimum of $B$. Assume the first alternative and denote $l = l^\pm$. The proof for the second alternative is analogous. Under this assumption we easily find

$$\xi_x(1) = \chi^- + (1 - \theta^-) = \chi^+ - (\theta^+ - 1)$$

Further on we denote

$$\mu = \inf\{B_z\colon l + \varepsilon < z < r^- - \varepsilon\}$$

$$\nu = \min\{\inf\{B_z\colon l - \varepsilon \leqslant z \leqslant l\}, \inf\{B_z\colon r^+ \leqslant z \leqslant r^+ + \varepsilon\}\}$$

Since there are no two local minima of $B_\cdot$ having the same hight and $l$ and $r^+$ are not local extrema of $B$, we have

$$\nu < h^* < \mu$$

We denote

$$\delta = \min\{\mu - h^*, h^* - \nu\} > 0$$

and prove that for any $\tilde{B} \in D(\mathbb{R})$

$$\sup\{|\tilde{B}_z - B_z|: z \in (l-\varepsilon, r^+ + \varepsilon]\} < \frac{\delta}{3} \Rightarrow |\tilde{\xi}_x(1) - \xi_x(1)| < 2\varepsilon \tag{21}$$

From where the assertion of the lemma follows.

Indeed, it is straightforward to check that with this choice of $\delta$ the assumption in (21) directly implies the following inequalities:

$$\min\{\tilde{B}_l, \tilde{B}_{r^-}\} < h^* + \frac{\delta}{2} < \inf\{\tilde{B}_z: z \in [l+\varepsilon, < r^- - \varepsilon]\}$$

$$\min\{\inf\{\tilde{B}_z: z \in [l-\varepsilon, l]\}, \inf\{\tilde{B}_z: z \in [r^+, r^+ + \varepsilon]\}\}$$
$$< h^* - \frac{\delta}{2} < \inf\{\tilde{B}_z: z \in [l, r^+]\}$$

From the first set of inequalities it follows that

$$\tilde{l}^- := \tilde{l}_x\left(h^* + \frac{\delta}{2}\right) \in [l, l+\varepsilon]$$

$$\tilde{r}^- := \tilde{r}_x\left(h^* + \frac{\delta}{2}\right) \in [r^-\varepsilon, r^-]$$

Similarly, from the second set of inequalities we get:

$$\tilde{l}^+ := \tilde{l}_x\left(h^* - \frac{\delta}{2}\right) \in [l-\varepsilon, l]$$

$$\tilde{r}^+ := \tilde{r}_x\left(h^* - \frac{\delta}{2}\right) \in [r^+, r^+ + \varepsilon]$$

Denoting

$$\tilde{\theta}^\pm := \tfrac{1}{2}(\tilde{r}^\pm - \tilde{l}^\pm)$$
$$\tilde{\chi}^\pm := \tfrac{1}{2}(\tilde{r}^\pm + \tilde{l}^\pm)$$

by the definition of the particle trajectories (8) we have

$$\tilde{\xi}_x(\tilde{\theta}^-) = \tilde{\chi}^- \quad \text{and} \quad \tilde{\xi}_x(\tilde{\theta}^+) = \tilde{\chi}^+$$

Clearly, $\tilde{\theta}^- \leqslant \theta^-$ and $\tilde{\theta}^+ \geqslant \theta^+$, so we have $\tilde{\theta}^- < 1 < \tilde{\theta}^+$. From Lipschitz-continuity of the trajectory $\tilde{\xi}_x(t)$ it follows that

$$\tilde{\chi}^+ - (\tilde{\theta}^+ - 1) \leqslant \tilde{\xi}_x(1) \leqslant \tilde{\chi}^- + (1 - \tilde{\theta}^-)$$

Putting all the ingredients together we find

$$(\tilde{\theta}^+ - \theta^+) + (\tilde{\chi}^+ - \chi^+) \leqslant \tilde{\xi}_x(1) - \xi_x(1) \leqslant (\theta^- - \tilde{\theta}^-) + (\tilde{\chi}^- - \chi^-)$$

From the obvious bounds $|\theta^\pm - \tilde{\theta}^\pm| \leqslant \varepsilon$, $|\chi^\pm - \tilde{\chi}^\pm| \leqslant \varepsilon$ finally we get

$$|\tilde{\xi}_x(1) - \xi_x(1)| \leqslant 2\varepsilon \quad \blacksquare$$

*Proof of Theorem 13.* We shall first show that the finite-dimensional distributions of $\hat{\eta}$ converge under the scaling to those of $\eta$:

$$\left\{\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n}\right\}_{x \in X,\, t \in T} \underset{n \to \infty}{\Rightarrow} \{\eta_x(t)\}_{x \in X,\, t \in T} \tag{22}$$

for any finite sets $X = \{x_1,..., x_k\} \subset \mathbb{R}$ *and* $T = \{t_1,..., t_l\} \subset \mathbb{R}^+$.

Let us extend the definition of $\xi_x(t)$ (8), and hence that of $\hat{\eta}_x(t)$ (15), to all $x \in \mathbb{R}$ by

$$\xi_x(t) = \xi(\Phi, x, t) = \xi(\Phi, \sup\{y < x : \Phi(y) \neq \Phi(x)\}, t),$$
$$\Phi \in D(\mathbb{R}), \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+$$

For CIG it implies that

$$\hat{\eta}_x(t) = \hat{\eta}_{\lfloor x \rfloor}(t), \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+$$

Note that

$$\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n} = \frac{\xi(\hat{\Phi}_0, \lfloor nx \rfloor, nt)}{n} = \frac{\xi(\hat{\Phi}_0, nx, nt)}{n} = \xi(\hat{\Phi}_0^{(n)}, x, t)$$

Now the scaling limit (22) follows from Donsker's theorem (ref. 3, p. 151) i.e.,

$$\hat{\Phi}_0^{(n)}(\cdot) \underset{n \to \infty}{\Rightarrow} B$$

and from the $B$-a.s. continuity of the functional $\xi(\cdot, x, t) : D(\mathbb{R}) \to \mathbb{R}$ (Lemma 14), by ref. 6, Theorem 6.7 on p. 365.

Now, let us note that, for any fixed $x \in \mathbb{R}$, $k > 0$, the sequence of probability measures

$$\left\{ P\left( \left\{ \frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n} \right\}_{t \in [0, k]} \right) \right\}_{n \geq 1}$$

is tight, since the trajectories are Lipschitz-continuous of order 1 (11) and uniformly bounded in $n$. By Corollary 7 in ref. 15, this tightness and (22) are enough to prove Theorem 13. ∎

Let us denote, as in the previous section, the coalescence time of CFP trajectories by $T_{x,y} = T_{x,y}(B)$, and that of CIG trajectories by $\hat{T}_{x,y} = T_{x,y}(\hat{\Phi})$. As one can expect from Theorem 13, an invariance principle holds for the coalescence times also:

**Theorem 15.** For any $x, y \in \mathbb{R}$:

$$\frac{\hat{T}_{\lfloor nx \rfloor, \lfloor ny \rfloor}}{n} \underset{n \to \infty}{\Rightarrow} T_{x,y}$$

This theorem can be proven along similar lines as Theorem 13. We safely leave it to the reader.

Recall that the explicit law of $T_{x,y}$ was derived in Proposition 12.

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